

# On the Symmetries of the Edgar-Ludwig Metric

Alan Barnes

School of Engineering and Applied Science, Aston University, Aston  
Triangle, Birmingham B4 7ET, UK †

PACS number(s): 04.20

Submitted to Class. Quantum Grav.

Date: 7 February 2008

**Abstract.** The conformal Killing equations for the most general (non-plane wave) conformally flat pure radiation field are solved to find the conformal Killing vectors. As expected fifteen independent conformal Killing vectors exist, but in general the metric admits no Killing or homothetic vectors. However for certain special cases a one-dimensional group of homotheties or motions may exist and in one very special case, overlooked by previous investigators, a two-dimensional homothety group exists. No higher dimensional groups of motions or homotheties are admitted by these metrics.

† E-Mail: barnesa@aston.ac.uk Fax: +44 121 333 6215

## 1. Introduction

The most general conformally flat pure radiation (or null fluid) field which is not a plane wave is given by (Ludwig and Edgar, 2000)

$$ds^2 = (xV(x, y, u) - w^2)du^2 + 2xdudw - 2wdudx - dx^2 - dy^2 \quad (1)$$

with

$$V = N(u)(x^2 + y^2) + 2M(u)x + 2F(u)y + 2S(u) \quad (2)$$

where  $M$ ,  $N \neq 0$ ,  $F$  and  $S$  are arbitrary functions of the coordinate  $u$ . This metric generalises a solution found by Wils (1989). Only three of these functions are essential as the form of the metric is preserved by the coordinate transformations

$$du = d\tilde{u}/\alpha(\tilde{u}) \quad w = \alpha\tilde{w} + \dot{\alpha}x \quad \tilde{x} = x \quad \tilde{y} = y \quad (3)$$

where  $\alpha$  is an arbitrary non-zero function of  $\tilde{u}$  and a dot signifies a partial derivative with respect to  $\tilde{u}$ . Under this coordinate transformation  $V$  transforms as

$$\tilde{V}(\tilde{x}, \tilde{y}, \tilde{u}) = \alpha^{-2}(V(x, y, u) + 2x(\alpha\ddot{\alpha} - \dot{\alpha}^2)) \quad (4)$$

In the original form of the metric (Edgar and Ludwig, 1997), this transformation was (in effect) used to set  $M = 0$ . However it is more convenient to use the coordinate freedom to set  $N = 1$  as in Edgar and Vickers (1999). Thus, dropping tildes, the most general conformally flat pure radiation field is given by the metric (1) with

$$V = x^2 + y^2 + 2M(u)x + 2F(u)y + 2S(u) \quad (5)$$

where  $M$ ,  $F$  and  $S$  are arbitrary functions of the coordinate  $u$ .

It is interesting to study the symmetries of the Edgar-Ludwig metric for a number of reasons. Firstly the metric is of interest in its own right as it has a physically realistic matter content, namely pure radiation. Being conformally flat the metric admits a fifteen parameter group of conformal symmetries and it might be thought that, for special values of the functions  $M$ ,  $F$  and  $S$ , some of these symmetries would reduce to homotheties or isometries as happens in the case of the conformally flat perfect fluid solutions found by Stephani (1967). Although the most general Stephani metrics admit no isometries, there is a rich set of special cases which admit isometries; see Barnes and Rowlingson (1990) and Seixas (1992) for a treatment of the non-expanding case and Barnes (1998) for the expanding case. By way of contrast in the conformally flat null fluid case, the special cases admitting homotheties and isometries are much more restricted except for the case of plane waves. In this paper only the non-plane wave case will be considered as the symmetry structure of plane waves solutions is well-known (Ehlers and Kundt, 1962).

Secondly metrics such as the Stephani and Edgar-Ludwig solutions present a stiff test for implementations of the Karlhede algorithm (Karlhede, 1980; Karlhede and

MacCallum, 1982) as their classification requires the calculation of the third and fourth derivatives of the curvature tensor respectively (Bradley, 1986; Koutras A, 1992; Skea, 1997). In a recent paper Pollney et al. (2000) presented a classification of the Edgar-Ludwig metric using the GRTensor implementation of the Karlhede algorithm. However their results are not consistent with those obtained by Skea (1997) using the implementation of the same algorithm in CLASSI (Åman, 1987). GRTensor predicts a non-trivial isotropy group whereas CLASSI predicts no isometries at all. Moreover there appears to be a bug in the GRTensor output routines as evidenced by the form given for the real part of  $(D^2\Phi)_{4,4}$  in Pollney et al. (2000). Given this uncertainty it is useful to investigate the symmetry structure by an alternative method, namely the direct integration of conformal Killing's equations. Moreover this method unlike the standard version of the Karlhede algorithm, gives expressions for the conformal Killing vectors directly. The results on isometries obtained in this paper are consistent with those obtained by the CLASSI version of the Karlhede algorithm.

Thirdly Ludwig and Edgar (2000) have developed a new formalism for the investigation of the existence of homotheties and isometries. They have used this formalism to investigate the symmetries of their metric and they plan to extend the method to handle conformal Killing vectors. Their formalism works well when only a single homothetic or Killing vector exists. However, when several such vectors exist, its use is somewhat controversial as evidenced by the discussion following Edgar's talk on the method at the recent GR16 meeting in Durban. Thus it is useful to have available results on the symmetry structure of this metric obtained by more tried and tested methods.

## 2. Conformal Killing Vectors

The conformal Killing equations are  $\xi_{i;j} + \xi_{j;i} = 2\sigma g_{ij}$  or equivalently

$$g_{ij,k}\xi^k + g_{ik}\xi_{,j}^k + g_{kj}\xi_{,i}^k = 2\sigma g_{ij} \quad (6)$$

A straightforward but somewhat lengthy calculation reveals that the components of the conformal Killing vector  $\xi^i$  and the associated conformal factor must have the form

$$\xi^u = (a(x^2 + y^2) + by + dx + c)/x \quad (7)$$

$$\xi^w = 2aw^2 + w(2\beta y + \gamma + 2a_u x - d_u) + H(u, x, y) \quad (8)$$

$$\xi^x = (a(x^2 - y^2) - by - c)w/x + 2\beta xy + \gamma x + a_u(x^2 - y^2) - b_u y - c_u \quad (9)$$

$$\xi^y = (2ay + b)w + \beta(y^2 - x^2) + \gamma y + \epsilon + 2a_u xy + b_u x \quad (10)$$

$$\sigma = 2aw + 2a_u x + 2\beta y + \gamma \quad (11)$$

where  $a, b, c, d, \beta, \gamma$  and  $\epsilon$  are functions of the coordinate  $u$  and where  $u$  subscripts denote partial derivatives.

Equations (7)-(11) could be used to investigate the conformal symmetries of any metric of the form (1) as their validity does not depend on the precise form of the function  $V$  in the metric. The  $xu$ ,  $yu$  and  $uu$  components of the conformal Killing equations do depend on the precise form of  $V$  and remain to be satisfied. For the Edgar-Ludwig metric where  $V$  is given by equation (5), integration of these three equations for  $H$  gives

$$\begin{aligned} H(x, y, w) = & (a_{uu} - 2aM)(x^2 + y^2) + (b_{uu} - 2bM)y + c_{uu} - 2cM \\ & - \frac{1}{2}ax^3 - axy^2 - (aF + b/2)xy + kx \\ & - x^{-1}(\frac{1}{2}ay^4 + (aF + b/2)y^3 + (bF + aS + c/2)y^2 + (cF + bS)y + cS) \end{aligned} \quad (12)$$

where  $k$  is a function of  $u$  only. The functions  $\beta$ ,  $\gamma$ ,  $\epsilon$ ,  $a$ ,  $b$ ,  $c$ ,  $d$  and  $k$  must also satisfy the following linear system of differential equations:

$$2\beta_u = 2aF - b \quad (13)$$

$$\gamma_u = 2aS - c \quad (14)$$

$$\epsilon_u = bS - cF \quad (15)$$

$$2a_{uuu} = 2aM_u + 2\beta F + 4a_u M - 2d_u - \gamma \quad (16)$$

$$b_{uuu} = bM_u + 2\beta S - dF_u + 2b_u M - 2d_u F - \epsilon \quad (17)$$

$$c_{uuu} = cM_u - dS_u + 2Mc_u - 2Sd_u - F\epsilon + S\gamma \quad (18)$$

$$2d_{uu} = -6aS + 2bF - 3c - 2k \quad (19)$$

$$2k_u = -2aS_u - 2dM_u + 2a_u S - 2b_u F + c_u - 4d_u M \quad (20)$$

Using standard techniques this linear system may be reduced to a first order linear system for fifteen unknowns, namely  $a$ ,  $b$ ,  $c$  and their first and second derivatives,  $d$  and its first derivative together with  $\beta$ ,  $\gamma$ ,  $\epsilon$  and  $k$ . The general solution of this system thus involves fifteen arbitrary constants of integration and so there is a fifteen-parameter family of conformal Killing vectors. This is to be expected as the metric is conformally flat and so admits a conformal symmetry group of maximal dimension. The algebra involved in deriving equations (12)-(20) is rather heavy and has been checked<sup>‡</sup> using the computer algebra system Reduce (Hearn, 1995).

### 3. Homotheties and Isometries

We now consider whether the metric can admit any homothetic motions or isometries. For a proper homothetic motion, the conformal factor  $\sigma$  appearing in equation (6) must be a non-zero constant and for an isometry we have  $\sigma = 0$ . Hence from equation (11) it follows immediately that  $a = \beta = 0$  and  $\gamma = \sigma$ . Since  $\sigma$  is constant, equations (13)-(15) imply that  $b = c = \epsilon_u = 0$ . Furthermore from (16) and (17) it follows

<sup>‡</sup> For full details see the files <http://www.aston.ac.uk/~barnesa/el.red> and `el.log`.

that  $d = d_0 - \frac{1}{2}\sigma u$  and  $k = 0$  where  $d_0$  is a constant. From (12) it now follows that  $H(x, y, u) = 0$ .

Equations (17), (18) and (20) then restrict the form of the metric functions  $F(u)$ ,  $S(u)$  and  $M(u)$  as follows

$$(2d_0 - \sigma u)M_u = 2\sigma M \quad (21)$$

$$(2d_0 - \sigma u)F_u = 2\sigma F - 2\epsilon \quad (22)$$

$$(2d_0 - \sigma u)S_u = 4\sigma S - 2\epsilon F \quad (23)$$

where  $d_0$ ,  $\epsilon$  and  $\sigma$  are all constants. In general these equations will not be satisfied and so the generic metric will not admit any homotheties or isometries. However, in special cases  $F$ ,  $S$  and  $M$  will satisfy these equations and the metric will admit one (or more) homotheties or isometries.

For an isometry  $\sigma = 0$  and (since  $\xi^i \neq 0$ ) it follows that  $d_0 \neq 0$  and so without loss of generality we may rescale  $\xi^i$  so that  $d_0 = 1$ . The solution of equations (21)-(23) is

$$M(u) = m_0 \quad F(u) = f_0 - \epsilon u \quad S(u) = s_0 + \frac{1}{2}\epsilon^2 u^2 - f_0 \epsilon u \quad (24)$$

where  $m_0$ ,  $f_0$  and  $s_0$  are arbitrary constants. Without loss of generality we may set  $f_0 = 0$  by means of the coordinate transformation  $\tilde{u} = u - f_0/\epsilon$  if  $\epsilon \neq 0$  or  $\tilde{y} = y + f_0$  if  $\epsilon = 0$ . Thus the metric admits a Killing vector if and only if the function  $V$  in the metric (1) can be written as

$$V = x^2 + y^2 + 2m_0 x - 2\epsilon u y + 2s_0 + \epsilon^2 u^2 \quad (25)$$

The Killing vector has the form

$$\xi = \partial_u + \epsilon \partial_y \quad (26)$$

This result agrees with that obtained by Ludwig and Edgar (2000) and is consistent with that of Skea (1997) who worked in a coordinate system in which  $M = 0$  rather than  $N = 1$  in equation (2).

For an homothety we may scale  $\xi^i$  so that  $\sigma = 1$  and then, by the coordinate transformation  $\tilde{u} = u - 2d_0$ , we may set  $d_0 = 0$ . The solution of equations (21)-(23) in this case is

$$M = m_1 u^{-2} \quad F = f_1 u^{-2} + \epsilon \quad S = s_1 u^{-4} + f_1 \epsilon u^{-2} + \epsilon^2/2 \quad (27)$$

where  $f_1$ ,  $m_1$  and  $s_1$  are arbitrary constants. Without loss of generality we may set  $\epsilon = 0$  by means of the coordinate transformation  $\tilde{y} = y + \epsilon$ . Thus the metric admits a homothetic vector if and only if the metric function  $V$  in the metric (1) can be written as

$$V = x^2 + y^2 + 2m_1 u^{-2} x + 2f_1 u^{-2} y + 2s_1 u^{-4} \quad (28)$$

The homothetic vector has the form

$$\xi = -\frac{1}{2}u\partial_u + \frac{3}{2}w\partial_w + x\partial_x + y\partial_y \quad (29)$$

This result is also consistent with that obtained by Ludwig and Edgar (2000) although they used a different coordinate system from the one they used to investigate Killing vectors. In this coordinate system the function  $N$  appearing in  $V$  in equation (2) is not normalised to unity. The use of different coordinate systems for the investigation of homotheties and isometries makes it difficult to determine whether the spacetime can admit two (or more) Killing and homothetic vectors. However the analysis is straightforward if the same coordinate system is used throughout; equations (25) and (28) are satisfied simultaneously if and only if

$$V = x^2 + y^2 \quad (30)$$

and in this case the metric (1) admits a Killing vector  $\xi_0$  and a homothetic vector  $\xi_1$  given by

$$\xi_0 = \partial_u \quad \xi_1 = -\frac{1}{2}u\partial_u + \frac{3}{2}w\partial_w + x\partial_x + y\partial_y \quad (31)$$

In this case the (maximal) homothety group is two-dimensional. The group is non-Abelian as the commutator of its generators is

$$[\xi_0, \xi_1] = -\frac{1}{2}\xi_0 \quad (32)$$

This case was overlooked by Ludwig and Edgar (2000) who concluded that maximal dimension of the homothety group admitted by their metric was one.

#### 4. Summary

The conformal symmetries of the Edgar-Ludwig metric have been investigated and an explicit form for the most general conformal Killing vector obtained. This vector depends on fifteen functions of the coordinate  $u$  which satisfy a first order linear differential system. The most general Edgar-Ludwig metrics which admit a Killing vector or a homothetic vector have been obtained; they depend on three arbitrary constants ( $m_0$ ,  $s_0$  and  $\epsilon$  or  $m_1$ ,  $s_1$  and  $f_1$  respectively) whereas the general metric depends on three arbitrary functions of  $u$ . Finally it has been shown that there is a single metric, overlooked by previous investigators, which admits a two-dimensional homothety group.

#### References

- Barnes A and Rowlingson R R 1990 *Class. Quantum Grav.* **7** 1721–31  
 Barnes A 1998 *Class. Quantum Grav.* **15** 3061–70

- Bradley M 1986 *Class. Quantum Grav.* **3** 317–34
- Edgar S B and Ludwig G 1997 *Class. Quantum Grav.* **14** L65–8
- Edgar S B and Vickers J A 1999 *Class. Quantum Grav.* **16** 589–604
- Ehlers J and Kundt W 1962 *Gravitation: an introduction to current research*, ed. L Witten Wiley, New York
- Hearn A C 1995 *Reduce User's Manual, Version 3.6* Rand, Santa Monica, CA
- Karlhede A 1980 *Gen. Rel. Grav.* **12** 693–707
- Karlhede A and MacCallum M A H 1982 *Gen. Rel. Grav.* **14** 673–82
- Koutras A 1992 *Class. Quantum Grav.* **9** L143–5
- Ludwig G and Edgar S B 2000 *Class. Quantum Grav.* **17** 1683–1705
- Pollney D, Skea J E F and d'Inverno R A 2000 *Class. Quantum Grav.* **17** 2885–2902
- Seixas W 1992 *Class. Quantum Grav.* **9** 225–38
- Skea J E F 1997 *Class. Quantum Grav.* **14** 2393–2404
- Stephani H 1967 *Commun. Math. Phys.* **4** 137–42
- Wils P 1989 *Class. Quantum Grav.* **6** 1243–51
- Åman J E 1987 *Manual for CLASSI: classification program for geometries in general relativity*  
University of Stockholm, Institute of Theoretical Physics technical report